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Parametric Representations of Non-Steady One-Dimensional Flows

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1. INTRODUCTION

One reason for much of the successful mathematical development of the classical linear theories of mechanics and physics is the availability of some kinds of general representations for the solutions of the underlying partial differential equations, or the possession of versatile techniques for generating or discovering such representations. Conversely, most of the frustrations that hinder the mathematical development of modern nonlinear theories can be attributed to the lack of such representations or techniques. In this connection, the equations of fluid mechanics are notoriously intractable. Very little is known about the structure of their solutions that can be simply described or readily and profitably applied. Worse yet, these already formidable equations are now being further complicated by generalizing them to include chemical reactions and electromagnetic phenomena, or to deal with substances with unfamiliar equations of state. In the face of such rapidly deepening ignorance, it becomes imperative to seek new information concerning the general analytical structure of even comparatively simple classes of flows.

One of the simplest types of flow, which therefore possesses one of the most thoroughly investigated and most extensively developed theories, is nonsteady, inviscid, one-dimensional compressible flow. A survey of this subject has been made by Zaldastani [1] and a unified account has been given by von Mises *et al.* [2]. The general results about the analytical structure of one-dimensional flow are of very limited extent. Knowledge of parametric representations of general solutions of the partial differential equations, in terms of a known Riemann function, is restricted primarily to the isentropic flow of ideal gases or of gases with closely related equations of state determined by Sauer [3, 4]. As far as possibly anisentropic flows are concerned, the general solution has been constructed by Martin [5] and Ludford [6] for equations of state that include the Karman-Tsien approximation to an ideal gas.

Investigations in the theories of detonation and of hypervelocity impact have stimulated interest in certain one-dimensional flow problems that involve complicated empirical equations of state. In fact, computations of (at least) one-dimensional flows have been used in attempts to confirm or determine equations of state. It would be aesthetically desirable, of course, to eliminate the need for such computation, or at least to develop alternative techniques, for recomputing, interpreting, or organizing the results. Accordingly, it is natural to ask whether there is any prospect of exploiting existing mathematical theory to a greater extent than is customary in conventional calculations. The most promising and most unconventional approach would require the construction of a parametric representation of the flow. The determination of a general structure will clearly depend very strongly on the formulation of the equations. The traditional Eulerian of Lagrangian formulations have probably been thoroughly but fruitlessly examined by so many investigators, that there is scant hope that a possibility for a general representation has been overlooked. A less familiar and therefore more promising possibility is Martin's reformulation in terms of a special type of Monge-Ampère equation [7], in which the form of one coefficient depends on the form of the equation of state. Apparently all that has been accomplished up to the present time to exploit this formulation is Martin's and Ludford's determination [5, 6, 7] of all equations of state (or equivalent information) for which the associated Monge-Ampère equation will admit an intermediate integral. Except for the special case mentioned at the end of the preceding paragraph, the existence of intermediate integrals merely makes it possible to reduce the determination of the general solution to a matter of solving a linear partial differential equation.

In this paper we shall derive a general parametric representation for all one-dimensional flows. We begin by considering two distinct solutions of Martin's Monge-Ampère equation for the same arbitrary equation of state. We observe that they determine an area-preserving mapping of a region of one plane onto a region in another plane. A familiar parametric representation of such mappings involves an arbitrary function $H(\alpha, \beta)$. In our application it develops that we must have $\alpha = \partial z/\partial p$, $\beta = \partial z/\partial \psi$ for any solution $z(p, \psi)$ of the Euler equation $\partial H_\alpha/\partial p + \partial H_\beta/\partial \psi = 0$; incidentally, this can be linearized exactly by a Legendre transformation. Accordingly, if a particular solution of one of Martin's equations is known, any other solution can be derived therefrom with the aid of some solution of some appropriately chosen linear partial differential equation of the second order. We have shown how to construct the required particular solutions for extensive classes of equations of state, which include that of an ideal gas, as well as a set closely related to the set of all harmonic functions of p and ψ.

This paper is merely a preliminary presentation of a fundamental principle or method. No effort has been made to apply our results to the solution of boundary or initial value problems. Off-hand this will be extremely difficult. On the other hand, our results could be straightforwardly applied to the inverse problem of constructing vast numbers of exact solutions of one-dimensional flow problems, in the hope of revealing useful examples. In view of the complexity of our parametric representation, such an experimental course of action could only have been contemplated in the era of automatic computers, to be performed, preferably, on a system with an on-line visual display unit. It also seems worthwhile to consider the possibility of using the results of a sufficiently accurate conventional one-dimensional flow computation to try to determine the approximate functional form of $H(\alpha, \beta)$. If the associated Euler equation could be solved exactly, as in the examples exhibited in the sequel, we would then be able to concoct an accurate parametric representation of the previously computed flow. This would certainly lead to a remarkable result in practical approximation, or in the simultaneous fitting of several functions of two variables.

It should be remarked that our results can be extended to "frozen" one-dimensional flows by observing that each of the composition variables must be a function of the path function ψ. This opens the possibility of forming possibly explicitly solvable linearized equations for chemically reacting flows by perturbing one of our myriad of exactly parametrically describable "frozen" flows. It will also be shown in Section 7 that nonsteady one-dimensional magnetohydrodynamic flows can be treated by the methods of this paper.

2. Formulation in Terms of a Monge-Ampère Equation

In Eulerian form the equations of inviscid one-dimensional flow are

$$\rho(u_t + uu_x) + p_x = 0, \tag{2.1}$$

$$\rho_t + (\rho u)_x = 0, \tag{2.2}$$

$$s_t + us_x = 0. \tag{2.3}$$

Here x, t, u, p, ρ, and s have the usual significance of coordinate, time, fluid velocity, pressure, density, and specific entropy. Literal subscripts will be used to designate partial differentiation with respect to the corresponding argument. To produce a fully determined system we must supplement these equations by an equation of state,

$$p = p(\rho, s), \tag{2.4}$$

as well as by appropriate initial or boundary conditions, to be discussed later.

M. H. Martin [7, 2, pp. 231-233] has reformulated these equations in terms of the solution of a Monge-Ampère equation. Since this is vital to our discussion, we shall repeat his derivation. Equations (2.1) and (2.2) imply the existence of functions $\eta(x, t)$ and $\psi(x, t)$ such that

$$d\psi = \rho\, dx - \rho u\, dt \tag{2.5}$$

$$\begin{aligned} d\eta &= \rho u\, dx - (\rho u^2 + p)\, dt \\ &= u\, d\psi - p\, dt. \end{aligned} \tag{2.6}$$

Clearly, ψ is constant on the particle paths, defined by $dx = u\, dt$, and it also satisfies

$$\psi_t + u\psi_x = 0. \tag{2.7}$$

In a region where any two paths $\psi = c_1$ and $\psi = c_2$ for different constants c_1 and c_2 are distinct, we must have

$$s = s(\psi) \tag{2.8}$$

by (2.3) and (2.7). Now let

$$\xi = \eta + pt. \tag{2.9}$$

Then by (2.6) and (2.9)

$$d\xi = u\, d\psi + t\, dp. \tag{2.10}$$

First, let us suppose that p and ψ are functionally independent, as will generally be the case. Functionally dependent p and ψ will be discussed at the end of this section. If we choose p and ψ to be our independent variables, then by (2.5) and (2.10)

$$t = \xi_p, \qquad u = \xi_\psi, \tag{2.11}$$

$$x_p = \xi_\psi \xi_{pp}, \qquad x_\psi = \xi_\psi \xi_{p\psi} + \frac{1}{\rho}. \tag{2.12}$$

By differentiating to eliminate x from (2.12) we obtain

$$\xi_{pp}\xi_{\psi\psi} - \xi_{p\psi}^2 + A^2 = 0, \tag{2.13}$$

where

$$A^2 = -\left(\frac{1}{\rho}\right)_p. \tag{2.14}$$

Clearly $A^2 = \rho_p/\rho^2 = 1/\rho^2 p_\rho > 0$, since the squared speed of sound, p_ρ, is positive.

The description of a one-dimensional flow has been reduced to the determination of $s(\psi)$ and $\xi(p, \psi)$. When these have been found, t and u can be computed by differentiation, and x by a quadrature. Except for a translation corresponding to the constant of integration for x, the flow is uniquely determined.

For $A^2 > 0$ (2.13) is a Monge-Ampère equation of hyperbolic type. Later on we shall require the corresponding characteristic equations. Since a derivation requires very little space, we shall quickly sketch it, rather than merely quoting the results. Let primes denote derivatives along a characteristic. Then (2.11) yields

$$t' = \xi_{pp}p' + \xi_{p\psi}\psi', \qquad u = \xi_{p\psi}p' + \xi_{\psi\psi}\psi'. \tag{2.15}$$

If $\psi' \neq 0$, apply (2.13) to the result of eliminating p' from (2.15) to obtain

$$\xi_{pp}u' - \xi_{p\psi}t' = -A^2\psi'. \tag{2.16}$$

Then (2.15) and (2.16), viewed as linear equations for the second partial derivatives of ξ, will be dependent if and only if

$$\psi'(u'\psi' + t'p') = 0, \qquad \psi'(A^2\psi'^2 - t'^2) = 0.$$

These imply

$$t' = \mp Ap', \qquad u' = \pm A\psi'. \tag{2.17}$$

If $\psi' = 0$, $p' \neq 0$, elimination of p' leads again to (2.17). If $\psi' = p' = 0$, equations (2.17) are satisfied automatically.

If we supplement (2.17) by (2.10) and introduce characteristic variables a and b, we obtain, finally

$$\xi_a = tp_a + u\psi_a\,, \qquad \xi_b = tp_b + u\psi_b\,, \tag{2.18}$$

$$u_a = A(p, \psi)\,p_a\,, \qquad u_b = -A(p, \psi)\,p_b\,, \tag{2.19}$$

$$t_a = -A(p, \psi)\,\psi_a\,, \qquad t_b = A(p, \psi)\,\psi_b\,. \tag{2.20}$$

Now let us investigate the exceptional case of flows for which p and ψ are functionally dependent, a problem considered by Weir [9]. We may assume that ψ is not a function of t only since this would imply $\rho = \psi_x = 0$. Thus ψ is certainly nonconstant, so we may assume

$$p = p(\psi). \tag{2.21}$$

If we can solve (2.8) for $\rho = \rho(p, s)$ we have also

$$\rho = \rho(\psi). \tag{2.22}$$

Now by (2.2) and (2.7) $u_x = 0$, whence

$$u = u(t). \tag{2.23}$$

Then by (2.1) and (2.5) $u'(t) = -p'(\psi)$. Since ψ is not a function of t only, we must have $u'(t) = -p'(\psi) = K =$ constant. Thus

$$u(t) = Kt + L, \tag{2.24}$$

$$p(\psi) = M - K\psi, \tag{2.25}$$

where L and M are constants. Finally, by (2.7) and (2.24)

$$\psi = F(x - \tfrac{1}{2}Kt^2 - Lt), \tag{2.26}$$

where F is an arbitrary function of its argument. Incidentally, all paths are represented in the xt-plane by a family of parallel lines or by a family of congruent parabolas.

If (2.8) cannot be solved for ρ, we still have (2.21), of course. Now by (2.5) and (2.6)

$$\psi_t = -\rho u = -\eta_x .$$

Hence for some $w(x, t)$

$$\psi = w_x , \qquad \eta = -w_t . \tag{2.27}$$

Again, by (2.5), (2.6), and (2.27)

$$\rho = w_{xx} , \qquad \rho u = -w_{xt} , \qquad \rho u^2 + p = w_{tt} \tag{2.28}$$

If we eliminate ρ and u from (2.8) we obtain

$$w_{xx}w_{tt} - w_{xt}^2 = w_{xx}p(w_x). \tag{2.29}$$

Since by (2.28) the Jacobian

$$\frac{\partial(w_x , w_t)}{\partial(x, t)} = \rho^2 p > 0,$$

then w_x and w_t must be functionally independent. Now under the Legendre transformation

$$W(\psi, \eta) = x\psi - t\eta - w(x, t) \tag{2.30}$$

we have

$$W_\psi = x, \qquad W_\eta = -t \tag{2.31}$$

and (2.29) becomes

$$p(\psi)\, W_{\eta\eta} = -1.$$

Thus

$$W(\psi, \eta) = -\tfrac{1}{2} p^{-1}(\psi)\, \eta^2 + f(\psi)\, \eta + g(\psi) \tag{2.32}$$

for arbitrary f and g. By using this result successively in Eqs. (2.31) back to (2.27) we can complete the description of our flow.

3. Area-Preserving Mappings

The study of (2.13) will lead us to consider the equation

$$\frac{\partial(X, Y)}{\partial(x, y)} = 1. \tag{3.1}$$

Any solution $X(x, y)$, $Y(x, y)$ defines an area-preserving mapping of a region of the xy-plane onto another region of the XY-plane. Parametric representations of such mappings can be constructed as follows [10, 11]. Let us write

$$\begin{aligned} x &= x(\alpha, \beta), \qquad & y &= y(\alpha, \beta), \\ X &= X(\alpha, \beta), \qquad & Y &= Y(\alpha, \beta), \end{aligned} \tag{3.2}$$

and let us assume that (3.2) can be solved uniquely for $\alpha = \alpha(x, y)$, $\beta = \beta(x, y)$. Then (3.1) implies

$$\frac{\partial(x, y)}{\partial(\alpha, \beta)} = \frac{\partial(X, Y)}{\partial(\alpha, \beta)}. \tag{3.3}$$

Case 1. Let us assume that $x + X$ and $y + Y$ are functionally independent, and then let us first make the special choice of parameters

$$x + X = 2\alpha, \qquad y + Y = 2\beta.$$

These definitions are equivalent to the assertion that for some functions $F(\alpha, \beta)$ and $G(\alpha, \beta)$

$$\begin{aligned} x &= \alpha + F(\alpha, \beta), \qquad & y &= \beta + G(\alpha, \beta) \\ X &= \alpha - F(\alpha, \beta), \qquad & Y &= \beta - G(\alpha, \beta). \end{aligned}$$

These equations and (3.3) imply

$$F_\alpha + G_\beta = 0,$$

whence

$$F = H_\beta, \qquad G = -H_\alpha$$

for some $H(\alpha, \beta)$. Hence

$$x = \alpha + H_\beta, \qquad y = \beta - H_\alpha,$$
$$X = \alpha - H_\beta, \qquad Y = \beta + H_\alpha. \tag{3.4}$$

From the first pair of equations we obtain

$$\frac{\partial(x, y)}{\partial(\alpha, \beta)} = H_{\alpha\alpha} H_{\beta\beta} - H_{\alpha\beta}^2 + 1.$$

These manipulations have established

LEMMA 3.1. *If $H(\alpha, \beta)$ does not satisfy the Monge-Ampère equation*

$$H_{\alpha\alpha} H_{\beta\beta} - H_{\alpha\beta}^2 + 1 = 0, \tag{3.5}$$

then equations (3.4) define an area-preserving mapping for which $x + X$ and $y + Y$ are functionally independent.

Incidentally, for any solution of (3.5) $x = H_\beta$, $y = H_\alpha$ defines an area-preserving map of a region of the $\alpha\beta$-plane onto a region of the xy-plane. The general solution of (3.5) can easily be constructed by means of the characteristic equations (2.18) with $A = 1$ and appropriate changes of notation.

If we observe that under any one-to-one transformation

$$\alpha = \alpha(\alpha^*, \beta^*), \qquad \beta = \beta(\alpha^*, \beta^*), \tag{3.6}$$

we have

$$\frac{\partial(x, y)}{\partial(\alpha^*, \beta^*)} = \frac{\partial(x, y)}{\partial(\alpha, \beta)} \frac{\partial(\alpha, \beta)}{\partial(\alpha^*, \beta^*)}.$$

Thus the only effect of making the transformation (3.6) in (3.4) is to replace α and β in (3.3) by α^* and β^*. This proves

LEMMA 3.2. *The general parametric representation of an area-preserving mapping in which $x + X$ and $y + Y$ are functionally independent can be constructed by applying the transformation (3.6) to (3.4).* For the sake of clarity, we remark that the partial differentiations with respect to α and β must be performed before the substitution.

CASE 2. Next, let us assume that $x + X$ and $y + Y$ are functionally dependent. Then

$$\frac{\partial(x + X, y + Y)}{\partial(x, y)} = 0. \tag{3.7}$$

Now (3.1) and (3.7) imply

$$(x + X)_x + (y + Y)_y = 0.$$

Thus

$$x + X = H_y , \qquad y + Y = - H_x \tag{3.8}$$

for some $H(x, y)$. But, by hypothesis we must now have

$$F(H_x , H_y) = 0 \tag{3.9}$$

for some function F that is not identifically zero. Now let $H(x, y)$ be a solution of (3.9), and define X and Y by (3.8). A simple computation confirms that we indeed have

$$\frac{\partial(X, Y)}{\partial(x, y)} = 1 + \frac{\partial(H_x , H_y)}{\partial(x, y)} = 1.$$

Since (3.1) has been satisfied, this proves

LEMMA 3.3. *A necessary and sufficient condition that $X(x, y)$, $Y(x, y)$ define an area-preserving mapping in which $x + X$ and $y + Y$ are functionally dependent is that X and Y be of the form (3.8) for any solution $H(x, y)$ of any first order partial differential equation of the form (3.9). Incidentally, the surfaces $z = H(x, y)$ are developable.*

4. APPLICATION TO ONE-DIMENSIONAL FLOW

In the following sections we shall consider extensive classes of equations of state for which we can construct particular explicit solutions of the corresponding equation (2.13). In this section we shall assume that we know a solution and shall attempt to deduce the general solution from it. For this purpose, let $\xi(p, \psi)$ and $\xi^*(p, \psi)$ be two solutions of (2.13) for the same A. They define an area-preserving map of the ut-plane onto the u^*t^*-plane since by (2.13)

$$\frac{\partial(\xi_p , \xi_\psi)}{\partial(p, \psi)} = \frac{\partial(\xi_p{}^*, \xi_\psi{}^*)}{\partial(p, \psi)}, \tag{4.1}$$

and if we make a one-to-one transformation, $\alpha = \alpha(p, \psi)$, $\beta = \beta(p, \psi)$, we have

$$\frac{\partial(\xi_p , \xi_\psi)}{\partial(\alpha, \beta)} = \frac{\partial(\xi_p{}^*, \xi_\psi{}^*)}{\partial(\alpha, \beta)}.$$

CASE 1. Let us assume that $\xi_p + \xi_p{}^*$ and $\xi_\psi + \xi_\psi{}^*$ are functionally independent. In accordance with Lemma 3.1, we can choose α, β, and $H(\alpha, \beta)$ so that

$$\xi_p + \xi_p{}^* = 2\alpha, \qquad \xi_\psi + \xi_\psi{}^* = 2\beta, \tag{4.2}$$

$$\xi_p - \xi_p{}^* = 2H_\beta, \qquad \xi_\psi - \xi_\psi{}^* = -2H_\alpha. \tag{4.3}$$

By (4.2) $\alpha_\psi = \beta_p$, so for some function $z(p, \psi)$

$$\alpha = z_p, \qquad \beta = z_\psi \tag{4.4}$$

and then by (4.2)

$$\xi^* = -2z - \xi, \tag{4.5}$$

where a constant of integration has been absorbed in z. By (4.3) and (4.4) H and z satisfy

$$\frac{\partial}{\partial p}\frac{\partial}{\partial z_p} H(z_p, z_\psi) + \frac{\partial}{\partial \psi}\frac{\partial}{\partial z_\psi} H(z_p, z_\psi) = 0 \tag{4.6}$$

Note that (4.6) is the Euler equation corresponding to the integral

$$I = \iint H(z_p, z_\psi)\, dp\, d\psi. \tag{4.7}$$

In expanded form (4.6) becomes

$$H_{z_p z_p} z_{pp} + 2H_{z_p z_\psi} z_{p\psi} + H_{z_\psi z_\psi} z_{\psi\psi} = 0. \tag{4.8}$$

That the variational equation (4.6) has emerged during the course of our discussion is not particularly surprising. It is well known [11] that (2.13) is the Euler equation for

$$J = \iint [\xi_\psi{}^2 \xi_{pp} - 2\xi_\psi \xi_p \xi_{\psi p} + \xi_p{}^2 \xi_{\psi\psi} - 6A^2(p, \psi)\, \xi]\, dp\, d\psi.$$

If H is chosen arbitrarily, then (4.8) is a quasilinear partial differential equation that defines a class of corresponding functions $z(p, \psi)$. Once a solution z has been determined, (4.5) defines a new solution ξ^*. By varying the choice of H and of the solution z of (4.8) we can, in principle, generate all solutions ξ^* of (2.13) such that $(\xi + \xi^*)_p$ and $(\xi + \xi^*)_\psi$ are functionally independent. The potential value of this result may be increased by discussing the nature of the solutions of (4.8). Since α and β, as defined by (4.2) or by (4.4), are functionally independent, let us make the Legendre transformation

$$Z(\alpha, \beta) = p\alpha + \psi\beta - z(p, \psi) \tag{4.9}$$

with new independent variables α and β. Then

$$Z_\alpha = p, \qquad Z_\beta = \psi, \tag{4.10}$$

and (4.8) is transformed into the *linear* equation

$$H_{\beta\beta}Z_{\alpha\alpha} - 2H_{\alpha\beta}Z_{\alpha\beta} + H_{\alpha\alpha}Z_{\beta\beta} = 0. \tag{4.11}$$

If we wished to solve a particular flow problem, say an initial-value problem we would have to determine the appropriate H and Z. If we are content to solve the "inverse" problem, which amounts to constructing examples of solutions of the equations of one-dimensional flow, it will suffice to choose functions H for which particular solutions or even the general solution of (4.11) can be found. The following are three obvious possibilities:

$$\begin{aligned} H &= \alpha\beta, & Z &= F(\alpha) + G(\beta), \\ H &= \tfrac{1}{2}(\alpha^2 - \beta^2), & Z &= F(\alpha + \beta) + G(\alpha - \beta), \\ H &= \tfrac{1}{2}(\alpha^2 + \beta^2), & Z &= \operatorname{Re} K(\alpha + i\beta), \end{aligned} \tag{4.12}$$

where F and G are arbitrary functions of their arguments, and K is an arbitrary analytic function of the complex variable $\alpha + i\beta$. It should also be remarked that for the choice

$$H(\alpha, \beta) = (1 + \alpha^2 + \beta^2)^{1/2} \tag{4.13}$$

(4.8) becomes the equation of minimal surfaces,

$$(1 + \beta^2)\, z_{pp} - 2\alpha\beta z_{p\psi} + (1 + \alpha^2)\, z_{\psi\psi} = 0. \tag{4.14}$$

The classical Weierstrass formulas

$$p = \operatorname{Re} \int (\Phi^2 - \Psi^2)\, dw, \qquad \psi = \operatorname{Re} \int i(\Phi^2 + \Psi^2) dw,$$

$$z = \operatorname{Re} \int 2\Phi\Psi\, dw \tag{4.15}$$

will lead to a parametric representation that depends on two arbitrary analytic functions of the complex variable w. Note also that the analog of (4.11) for (4.14) is

$$(1 + \alpha^2)\, Z_{\alpha\alpha} + 2\alpha\beta Z_{\alpha\beta} + (1 + \beta^2)\, Z_{\beta\beta} = 0,$$

an equation obtained when we seek functions homogeneous of order one,

$$G(A, B, C) = CZ(\alpha, \beta),$$

with $\alpha = A/C$, $\beta = B/C$, that satisfy Laplace's equation

$$G_{AA} + G_{BB} + G_{CC} = 0.$$

If we set $C = iD$ in Laplace's equation we obtain the two-dimensional wave equation

$$G_{AA} + G_{BB} - G_{DD} = 0.$$

The search for solutions that are homogeneous of order one, familiar from the theory of linearized conical flow [8] again leads via the Weierstrass formulas for minimal surfaces to a parametric representation of the solutions of

$$(1 - \beta^2) z_{xx} + 2\alpha\beta z_{xy} + (1 - \alpha^2) z_{yy} = 0,$$

which corresponds to the choice

$$H(\alpha, \beta) = (1 - \alpha^2 - \beta^2)^{1/2}.$$

We may also speculate that in the equations (4.11) there should be a fairly extensive subset that can be reduced by simple explicit transformations to the canonical form

$$\frac{\partial^2 Z}{\partial \alpha^{*2}} + \frac{\partial^2 Z}{\partial \beta^{*2}} + B(\alpha^*, \beta^*) \frac{\partial Z}{\partial \alpha^*} + C(\alpha^*, \beta^*) \frac{\partial Z}{\partial \beta^*} = 0.$$

Among these transformed equations there should even be a fairly extensive subset for which Bergman's integral operators [12] can easily be constructed. An application of one of these operators to any analytic function of an appropriate complex variable would then yield a permissible Z.

Perhaps the simplest way to generate solutions of (2.13) depends on the fact that (4.11) remains unchanged if we interchange H and Z. Now suppose we have chosen an arbitrary H and consider some solution Z of (4.11). Without actually knowing Z we can make the new choices

$$H'(\alpha, \beta) = Z(\alpha, \beta), \qquad Z'(\alpha, \beta) = H(\alpha, \beta).$$

Then in accordance with (4.5), (4.9), and (4.10) we obtain

$$p = Z_\alpha' = H_\alpha, \qquad \psi = Z_\beta' = H_\beta,$$
$$z' = \alpha p + \beta\psi - Z' = \alpha H_\alpha + \beta H_\beta - H$$
$$\xi' = 2z' - \xi.$$

Thus, if we eliminate α and β to express z' as a function of p and ψ, then we shall have constructed a solution ξ' of (2.13) that depends on an arbitrary function H and its derivatives.

The result just obtained suggests the following process. First select a particular solution $\xi_0(p, \psi)$ of (2.13) and an arbitrary function $H_1(\alpha, \beta)$. Construct $z_1(p, \psi)$ by the process just described, and form

$$\xi_1(p, \psi) = 2z_1(p, \psi) - \xi_0(p, \psi),$$

which also satisfies (2.13). Now select another arbitrary $H_2(\alpha, \beta)$, construct a corresponding $z_2(p, \psi)$ and form

$$\xi_2(p, \psi) = 2z_2 - \xi_1 = 2z_2(p, \psi) - 2z_1(p, \psi) + \xi_0(p, \psi).$$

Since this is a solution of (2.13) that depends on two arbitrary functions, one would expect that this provides sufficient generality to solve the Cauchy problem.

The flow corresponding to ξ^* can easily be described in terms of the parameters α and β. First note that p and ψ are defined by (4.10). From (2.11), (4.4), and (4.5) we obtain

$$\begin{aligned} t^* &= \xi_p{}^* = z_p - \xi_p = \alpha - t(Z_\alpha, Z_\beta), \\ u^* &= \xi_\psi{}^* = z_\psi - \xi_\psi = \beta - u(Z_\alpha, Z_\beta), \end{aligned} \tag{4.16}$$

and from (2.12)

$$\begin{aligned} dx^* &= \xi_\psi{}^* \, d\xi_p{}^* + \rho^{-1} \, d\psi \\ &= (\beta - u) \, [d\alpha - \xi_{pp} \, dZ_\alpha - \xi_{p\psi} \, dZ_\beta] + \rho^{-1} \, dZ_\beta . \end{aligned} \tag{4.17}$$

From (4.17) we can easily compute the line integral

$$x^* = \int x_\alpha{}^* \, d\alpha + x_\beta{}^* \, d\beta. \tag{4.18}$$

Later, in Case 2, we shall show that these parametric representations describe all one-dimensional flows for which p and ψ are independent with minor exceptions. However, it may be instructive and reassuring to examine the question of the generality of our representation on the following heuristic basis. A typical initial value problem in one-dimensional flow can be phrased as follows. Determine $u(x, t)$, $\rho(x, t)$, and $s(x, t)$ so that on the prescribed initial curve

$$x = X(\sigma), \qquad t = T(\sigma), \tag{4.19}$$

these functions will assume the prescribed initial values

$$\begin{aligned} u(X, T) &= U(\sigma), \\ \rho(X, T) &= R(\sigma), \\ s(X, T) &= S(\sigma). \end{aligned} \tag{4.20}$$

In accordance with (2.5), from

$$\frac{d\Psi(\sigma)}{d\sigma} = \frac{R \, dX}{d\sigma} - \frac{RU \, dT}{d\sigma}$$

we can determine the initial function $\psi(X, T) = \Psi(\sigma)$ uniquely except for an unimportant constant of integration. On segments where $\Psi = \Psi(\sigma)$ is monotone we can invert uniquely to obtain $\sigma = \sigma(\Psi)$. Then the function $s(\psi)$ required in (2.8) is specified by

$$s(\Psi) = S(\sigma(\Psi)).$$

We can also readily find the initial pressure

$$P(\sigma) = p(R(\sigma), S(\sigma)).$$

If we eliminate σ between the four functions X, T, U, R, we obtain three relations that have to be satisfied simultaneously, one of which is simply an equation of the initial curve in the XT-plane. On the other hand, the specification of a particular equation (4.8) or (4.11) involves one arbitrary function H. If (4.11) is of hyperbolic type, then a typical initial value problem for that equation would involve two additional arbitrary functions. Thus we would find that in a very complicated way our parametric representation would depend on three arbitrary functions. Accordingly, we may hope that they will provide sufficient freedom to enable us to satisfy the three relations between the initial data. In this connection we remark that the choices (4.12), (4.15), etc. may generate extensive classes of flows, but certainly not the most general one-dimensional flow.

It will probably be an extremely difficult problem to determine the form of H required for a given boundary or initial value problem. However, as suggested in the Introduction, we may be able to exploit our ideas to interpret the results of an accurate conventional flow calculation. Let us assume that ξ^* is our computed solution. Let us assume that ξ is another computed solution or a solution obtained by methods similar to those to be discussed in the following sections. Then α, β, H_α, and H_β can be determined from (2.11), (4.2), and (4.3) expressed as

$$2\alpha = t^* + t, \qquad 2\beta = u^* + u,$$
$$2H_\beta = t^* - t, \qquad -2H_\alpha = u^* - u.$$

If we have sufficiently abundant results we may be able to determine an approximate functional form for H. If we also calculate z by (4.5) we could use (4.9) to (4.11) to check or improve our computation of H.

In view of the amount of knowledge of one-dimensional flow that we have accumulated, it might be worthwhile to remark that any one-dimensional flow can be converted into a slightly three-dimensional flow by adding two arbitrary cartesian velocity components of the form $v = v(\psi)$, $w = w(\psi)$.

CASE 2. Next, suppose $(\xi + \xi^*)_p$ and $(\xi + \xi^*)_\psi$ are functionally dependent. By Lemma 3.3 we must have

$$\xi_p^* + \xi_p = \frac{\partial H(\xi_p, \xi_\psi)}{\partial \xi_\psi},$$

$$\xi_\psi^* + \xi_\psi = -\frac{\partial H(\xi_p, \xi_\psi)}{\partial \xi_p}, \tag{4.21}$$

where H is a solution of some equation

$$F\left(\frac{\partial H}{\partial \xi_\psi}, -\frac{\partial H}{\partial \xi_p}\right) = 0. \tag{4.22}$$

From (4.21) we obtain

$$F(\xi_p{}^* + \xi_p, \xi_\psi{}^* + \xi_\psi) = 0. \tag{4.23}$$

As a first order partial differential equation satisfied by the solution ξ^* of (2.13), Eq. (4.23) defines an intermediate integral of (2.13).

Ludford [6] has shown that there are only two possible general functional forms for $A(p, \psi)$ for which (2.13) will admit intermediate integrals. One of them can be rejected immediately, since by contrast with (4.23) its intermediate integrals depend explicitly on ξ^*. The remaining possibility is

$$A(p, \psi) = G'(w) = \frac{dG}{dw}, \qquad w = \alpha p + \beta\psi. \tag{4.24}$$

If $\alpha^2 + \beta^2 \neq 0$, the desired intermediate integral is

$$\beta\xi_p{}^* - \alpha\xi_\psi{}^* \pm G(\alpha p + \beta\psi) = 0. \tag{4.25}$$

This can be easily confirmed. If we differentiate (4.25) we obtain

$$\xi^*_{pp}\beta - (\xi^*_{p\psi} \mp G')\,\alpha = 0,$$
$$(\xi^*_{p\psi} \pm G')\,\beta - \xi^*_{\psi\psi}\alpha = 0.$$

Since the determinant of the coefficients of α and β vanishes, ξ^* satisfies (2.13) with $A = G'$. Now note that since $\xi_p{}^*$ and $\xi_\psi{}^*$ appear in the intermediate integral (4.23) only in the combinations $(\xi^* + \xi)_p$ and $(\xi^* + \xi)_\psi$, (4.25) implies

$$\beta\xi_p - \alpha\xi_\psi \mp G(\alpha p + \beta\psi) = 0. \tag{4.26}$$

The general solution of (4.26),

$$\xi = K(\alpha p + \beta\psi) \pm \frac{(\alpha\psi - \beta p)}{\alpha^2 + \beta^2}\, G(\alpha p + \beta\psi), \tag{4.27}$$

depends on a single arbitrary function K. On the other hand, the general solution of (2.13) depends on *two* arbitrary functions. Thus the occurrence of Case 2 for $A = G'(\alpha p + \beta\psi)$ is exceptional. Accordingly, the processes described in Case 1 of this section will be required to produce the general solution.

If $\alpha = \beta = 0$, then $A = G'(0) =$ constant. Since the general integral of (2.13) can easily be derived directly from the characteristic equations (2.18)-(2.20) when $A =$ const [5], we shall not exhibit the intermediate integrals.

5. Isentropic Flows

For a general equation of state we have by (2.8) and (2.14)

$$A^2(p, \psi) = -\frac{\partial \rho^{-1}(p, s(\psi))}{\partial p}. \tag{5.1}$$

In isentropic flow s is constant, so

$$A = A(p) \tag{5.2}$$

is independent of ψ. The application of the methods of Section 4 for functions A of the form (5.2) can be treated as a special case of the discussion of the more general form (6.4) in the next section. However, we shall consider isentropic flow separately on account of its historical interest, and also because we wish to bring out the contrast between our approach and the application of characteristic equations, which can be carried farther for (5.2) than for (6.4). By (2.9) we can define the characteristic variables (except in simple waves) so that

$$u + P(p) = 2a, \qquad u - P(p) = 2b, \tag{5.3}$$

where

$$P(p) = \int A(p)\, dp. \tag{5.4}$$

Thus

$$u = a + b, \qquad P(p) = a - b, \tag{5.5}$$

whence

$$p = p(a - b). \tag{5.6}$$

By (5.2) and (5.6)

$$A(p) = A^*(a - b). \tag{5.7}$$

If we eliminate t or ψ from (2.20) we obtain

$$\psi_{ab} = \tfrac{1}{2}(\log A^*(a-b))'\,(\psi_a - \psi_b), \tag{5.8}$$

$$t_{ab} = -\tfrac{1}{2}(\log A^*(a-b))'\,(t_a - t_b). \tag{5.9}$$

Of course it is a classical result, quoted in most fluid dynamics texts, that the equations of one-dimensional isentropic flow can be linearized by the introduction of characteristic variables.

Sauer [3, 4] has determined forms of the equation of state (and hence of A^*) for which the Riemann function of (5.9) can be constructed explicitly in terms of classical transcendental functions. For the corresponding equations (5.9) we can therefore determine by quadratures the general solution, depending on two arbitrary functions. For other forms of A^* we can only construct particular solutions, at present. Thus it becomes attractive to contemplate using the methods developed in the preceding section to generate additional solutions.

An extensive set of solutions of the desired type has been given already in (4.27), of course. However, since the usefulness of the inverse method will be enhanced by finding as many solutions as possible, we shall consider some additional possibilities. If we observe that the combination $\psi_a - \psi_b$ vanishes for any function of $a + b$, we may be led to try

$$\psi = F(a - b) + G(a + b) \tag{5.10}$$

If this is to satisfy (5.8) we must have

$$\begin{aligned} G(a + b) &= c(a + b)^2 + d(a + b) + e, \\ F(a - b) &= B + 2c \int_C^{a-b} A^{*-1}(\tau) \int_C^{\tau} A^*(\sigma)\, d\sigma\, dt, \end{aligned} \tag{5.11}$$

where B, C, c, d, e are constants. From (2.18) and (2.20) we can obtain ξ and t by quadratures. From $a - b = P(p)$ and (5.11) we can solve for $a + b$ as a function of $\psi - F(P(p))$ if $c \neq 0$ or $d \neq 0$. Thus we can determine

$$a = a(p, \psi), \qquad b = b(p, \psi),$$

and then $\xi(a, b)$ can be transformed into a function of p and ψ. Accordingly, we can always find a solution of (2.13) .

We can also seek product solutions

$$\psi = F(x)\, G(y) \tag{5.12}$$

of (5.8), where

$$x = a + b, \qquad y = a - b.$$

Now (5.8) becomes

$$\psi_{xx} - \psi_{yy} = \tfrac{1}{4} (\log A^*(y))' \, \psi_y \, ,$$

whence

$$\frac{F''}{F} = \frac{[G'' + \frac{1}{4} (\log (A^*))']}{G} = \pm k^2,$$

for some constant k. If $G' \neq 0$, then

$$x = a - b = P(p) \qquad \text{and} \qquad \psi = F(P(p))\, G(a + b)$$

again determine a and b as functions of p and ψ. As before, we can determine $\xi(p, \psi)$ eventually by quadratures.

A more direct way to find a solution of (2.13) when $A = A(p)$ would be to seek a separable solution

$$\xi = f(p)\,\psi \tag{5.13}$$

of (2.13). We find that we must have

$$f(p) = \pm P(p),$$

where $P(p)$ is defined by (5.4). Now (2.11) and (2.12) yield

$$u = f(p), \qquad t = f'(p)\,\psi,$$
$$x = \left[ff' + \frac{1}{\rho}\right]\psi + B, \tag{5.14}$$

where B is a constant.

By the methods of the preceding section, we can start from these particular solutions to generate additional solutions. For this purpose we can use at least the choices (4.12), (4.15), etc.

6. Anisentropic Flows

M. H. Martin and G. S. S. Ludford [5, 6, 13] have considered the problem of determining intermediate integrals for (2.13). In other words, they have sought first order partial differential equations of the form

$$B(p, \psi, \xi, \xi_p, \xi_\psi) = 0, \tag{6.1}$$

which imply (2.13). Ludford showed that B exists if and only if A is of one of the forms

$$A = F(c_1 p + c_2 \psi), \tag{6.2}$$

or

$$A = \frac{1}{(p + c_1)(\psi + c_2)} F\left(\frac{p + c_1}{\psi + c_2}\right), \tag{6.3}$$

where c_1 and c_2 are constants, and F is an arbitrary function of its argument. If

$$A = (c_1 p + c_2 \psi + c_3)^{-2},$$

where c_3 is a constant, there exist two functionally independent intermediate integrals, and the general integral ξ of (2.13) can be constructed parametrically in terms of two arbitrary functions of one variable. To any other choice of A of the forms (6.2) or (6.3) there corresponds only one intermediate integral, from which an extensive class of particular solutions can be found. Now, however, determination of the general solution (2.13) can only be reduced to the solution of an appropriate linear partial differential equation.

Martin's earliest investigations of this problem were concerned with separable A's of the form

$$A(p, \psi) = E(p)\, G(\psi). \tag{6.4}$$

In accordance with (6.2) and (6.3) the only possibilities that will admit intermediate integrals are (i) $A = E(p)$; (ii) $A = G(\psi)$; and (iii) $A = (p + c_1)^{m\mp 1}/(\psi + c_2)^{m\pm 1}$ for some constant m. If we observe that for an ideal gas

$$\rho^{-1} = p^{-1/\gamma}\, e^{-s/c_p}\,,$$

and

$$A^2 = \gamma^{-1} p^{-1-1/\gamma}\, e^{-s/c_p},$$

then it will obviously be desirable to try to apply the process of Section 4 to produce anisentropic flows for choices of A of the form (6.4) that are more general than Martin's. The required particular solution can be chosen to be of the form

$$\xi = K(p) + L(\psi) \tag{6.5}$$

if in conformity with (2.13) and (6.4) we choose K and L to be solutions of

$$K''(p) = -\, M E^2(p),$$
$$L''(p) = \frac{G^2(\psi)}{M}\,, \tag{6.6}$$

for some constant $M \neq 0$.

Problems in the theory of detonation or of hypervelocity impact require equations of state more general than those of the form

$$\rho^{-1} = -\, G^2(\psi) \int E^2(p)\, dp + N(\psi),$$

for arbitrary $N(\psi)$, that lead to A of the form (6.4). In order to apply the results of Section 4 to these more general problems, we would require a particular solution of (2.13) for arbitrary $A(p, \psi)$. Since it is not clear how to find such a solution, the next best strategy would be to seek a rather extensive set $\{A_0(p, \psi)\}$ for each member of which a particular solution of (2.13) can be constructed, in the hope that (i) some $A_0(p, \psi)$ approximates $A(p, \psi)$ acceptably, and (ii)

$$\rho^{-1} = -\int A_0^2(p, \psi)\, dp + N(\psi), \qquad \psi = \psi(s)$$

approximates the equation of state satisfactorily.

In this connection, let $U(p, \psi)$ be a harmonic function, and let us determine the form of A such that $\xi = U$ satisfies (2.13). Let $V(p, \psi)$ be a harmonic conjugate of U. Then

$$f(\zeta) = U + iV$$

will be an analytic function of the complex variable

$$\zeta = p + i\psi.$$

Now

$$\frac{d^2f}{d\zeta^2} = f''(\zeta) = U_{pp} - iU_{p\psi} = - U_{pp} - iU_{p\psi}.$$

Thus U satisfies

$$U_{pp}U_{\psi\psi} - U_{p\psi}^2 + |f''(\zeta)|^2 = 0, \tag{6.7}$$

when $\xi = U$. Equation (6.9) merely asserts that $\log A_0(p, \psi)$ must be harmonic. Now if for a given equation of state the corresponding function $\log A(p, \psi)$ could be approximated acceptably by some harmonic function $\log A_0(p, \psi)$, this would determine $\operatorname{Re} \log f''(\zeta)$. Then it will be an easy matter to construct an $f(\zeta)$ from which to obtain $U = \operatorname{Re} f(\zeta)$.

In principle we could attempt to determine $A_0(p, \psi)$ as follows. Let B be some region of the ζ-plane in which a harmonic approximation to $\log A(p, \psi)$ is desired, and let C be its boundary. Let $\log A_0(p, \psi)$ be a solution of the interior Dirichlet problem for Laplace's equation, with boundary values $\log A_0 = \log A$ on C. At points ζ close enough to C this should yield a good approximation. In practice the success and effectiveness of this process would depend on the choice of regions B with known, easily described Green's functions. But this can be accomplished by use of a dictionary of conformal mappings.

7. Magnetohydrodynamic Flows

To show that nonsteady one-dimensional magnetohydrodynamic flows can be treated by the methods developed in the preceding sections it will suffice to reduce the discussion to the determination of a solution of (2.13). The equations to be solved are, for the case of infinite conductivity [14, 15],

$$\rho_t + (\rho u)_x = 0, \tag{7.1}$$

$$(\rho u)_t + \left(\rho u^2 + p + \frac{B^2}{2\mu}\right) = 0 \tag{7.2}$$

$$s_t + us_x = 0, \tag{7.3}$$

$$p = p(\rho, s), \tag{7.4}$$

$$B_t + (Bu)_x = 0, \tag{7.5}$$

where $B(x, t)$ is the magnetic induction, and μ (assumed constant) the permeability. As in Section 2 we can deduce from (7.1) and (7.2)

$$d\psi = \rho\, dx - \rho u\, dt, \tag{7.6}$$

$$\begin{aligned} d\eta &= \rho u\, dx - \left(\rho u^2 + p + \frac{B^2}{2\mu}\right) dt \\ &= u\, d\psi - \left(p + \frac{B^2}{2\mu}\right) dt \end{aligned} \tag{7.7}$$

for some ψ and η. The function ψ has the same significance as before, and we still have

$$s = s(\psi). \tag{7.8}$$

Also, by (7.1) and (7.5)

$$\left(\log \frac{B}{\rho}\right)_t + u \left(\log \frac{B}{\rho}\right)_x = 0.$$

Hence for some $w(\psi)$,

$$B = \rho w(\psi). \tag{7.9}$$

Now let us set

$$P = p + \frac{B^2}{2\mu} \tag{7.10}$$

and

$$\xi = \eta + Pt.$$

Then

$$d\xi = u\, d\psi + t\, dP, \tag{7.11}$$

which is formally identical with (7.10) except for a minor change in notation. By (7.4), (7.9), and (7.10)

$$P = P(p, \psi) = p(\rho, s(\psi)) + \frac{\rho^2 w^2(\psi)}{2\mu}. \tag{7.12}$$

Suppose P and ψ are functionally independent. Then from (7.6) and (7.11) we obtain (2.11) and (2.12) with p replaced by P. It is to be understood, also, that (7.12) has been inverted to determine $\rho = \rho(P, \psi)$. Then, exactly as before, we find that $\xi(P, \psi)$ must satisfy

$$\xi_{PP}\xi_{\psi\psi} - \xi_{P\psi}^2 + A^2(P, \psi) = 0,$$

where now

$$A^2(P, \psi) = -\left(\frac{1}{\rho}\right)_P .$$

From this point the results of Sections 3-6 can be applied immediately.

If P and ψ are functionally dependent, the discussion at the end of Section 2 can be repeated almost word for word if we replace p by P.

8. A Possible Extension to Cylindrically or Spherically Symmetrical Flows

In Eulerian form the equations of $(\epsilon + 1)$-dimensional spherically symmetrical motion consist of the equation of continuity

$$x^\epsilon \rho_t + (x^\epsilon \rho u)_x = 0, \tag{8.1}$$

supplemented by (2.1), (2.3), and (2.4). Now we can combine (8.1) and (2.1) to obtain

$$(x^\epsilon \rho u)_t + (x^\epsilon \rho u^2 + P)_x = 0 \tag{8.2}$$

where

$$P(x, t) = P(x_0, t) + \int_{x_0}^{x} x^\epsilon p_x \, dx, \tag{8.3}$$

for some constant x_0 and an arbitrary function $P(x_0, t)$ of t. By familiar arguments we can introduce $\psi(x, t)$ and $\eta(x, t)$ such that

$$d\psi = x^\epsilon \rho (dx - u \, dt), \tag{8.4}$$

$$\begin{aligned} d\eta &= x^\epsilon \rho u \, dx - (x^\epsilon \rho u^2 + P) \, dt \\ &= u \, d\psi - P \, dt. \end{aligned} \tag{8.5}$$

The function ψ is still constant on particle paths, so we still obtain

$$s = s(\psi). \tag{8.6}$$

If we set $\xi = \eta + Pt$ we again find

$$d\xi = d\eta + t \, dP. \tag{8.7}$$

If P and ψ are functionally independent we find

$$\xi_\psi = u, \qquad \xi_P = t, \tag{8.8}$$

$$x_\psi = \xi_\psi \xi_{P\psi} + \frac{1}{x^\epsilon \rho}, \qquad x_P = \xi_\psi \xi_{PP} . \tag{8.9}$$

In general $x^\epsilon \rho$ is an unknown function of P and ψ. However, let us continue our standard manipulations as if we knew this function. By adopting this attitude we should at least be able to adapt much of the formal development of Sections 3-7 to derive equations which could conceivably be exploited in the interpretation of experimental results or the fitting of computed flow fields.

By eliminating x from (8.9) we obtain the customary Monge-Ampère equation

$$\xi_{PP}\xi_{\psi\psi} - \xi^2_{P\psi} + A^2(P, \psi) = 0, \tag{8.10}$$

where

$$A^2 = -\left[\frac{x^{-\epsilon}(P, \psi)}{\rho(P, \psi)}\right]_P . \tag{8.11}$$

For $\epsilon = 1$ or 2 there is no *a priori* reason to expect $A^2 > 0$, but fortunately very little of our work in Sections 3–6 depends on the sign of A^2.

If we had sufficiently abundant quantities of empirical flow data, we should be able to approximate $\psi(x, t)$, $s(\psi)$, and $P(x, t)$. Then we should be able to approximate $x = x(P, \psi)$. If we also knew the equation of state, say in the form $\rho = \rho(p, s)$ we should be able to determine the approximate form of $x^\epsilon \rho$ as a function of P and ψ. If we optimistically use this to determine $A^2(P, \psi)$ by (8.11), we can now contemplate using the methods of Section 4 to attempt to determine an analytical approximation to ξ as a solution of (8.10). In accordance with our previous discussion, this would yield a parametric description of the flow.

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